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Relatively Best Chebyshev Approximations to exp(t)

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Let $R_{m,n}^{(-h,h)}(t)$ be the best (m, n) rational approximation to $\exp(t)$ over the interval $[-h, h]$ in the relative error norm, with largest relative error $\epsilon_{m,n}^{(-h,h)}$. In most of the following, the dependence on h is not used and we shall therefore omit the superscript.

Our main result is that $R_{m,n}^{(-h,h)}$ satisfies a symmetry property analogous to the Padé approximations to $\exp(t)$.

THEOREM. *If $R_{m,n}$ is nondegenerate, then $R_{n,m}$ is nondegenerate and the following relation holds*:

$$R_{n,m}(t)\, R_{m,n}(-t) = 1 - \epsilon_{m,n}^2 . \tag{1}$$

Proof. The approximation in the relative error norm is a special case of Chebyshev approximation with a positive weight function, and if $R_{m,n}$ is nondegenerate, the following alternation property holds (Achieser [1]): there exist $m + n + 2$ points $\{t_i\}_{i=0}^{m+n+1}$ such that $-h \leqslant t_0 < t_1 < t_2 < \cdots < t_{m+n+1} \leqslant h$ and $E(t_i) = (-1)^{i+k}\, \epsilon_{m,n}$, where k is 0 or 1 depending on the sign of $E(t_0)$ and E is the relative error function

$$E(t) \triangleq 1 - \exp(-t)\, R_{m,n}(t). \tag{2}$$

This property uniquely determines $R_{m,n}$.

We note that E' has at most $m + n$ zeros in $[-h, h]$, since E' is of the form $\exp(-t)\, Q(t)$, where Q is a rational function with a numerator of degree $m + n$. Therefore the $m + n + 2$ alternating points must be precisely these $m + n$ zeros $\{t_i\}_{i=1}^{m+n}$ together with $t_0 = -h$ and $t_{m+n+1} = h$.

We now consider the (n, m) rational function $S(t) = (1 - \epsilon_{m,n}^2)/R_{m,n}(-t)$, and find the extrema of its relative error function, i.e., the set

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$$\begin{aligned}
&\{t: (d/dt)(1 - \exp(-t)\, S(t)) = 0\}\\
&\quad = \{t: \exp(-t)(S(t) - S'(t)) = 0\}\\
&\quad = \{t: (1 - \epsilon_{m,n}^2)[1/R_{m,n}(-t) + R'_{m,n}(-t)/R_{m,n}^2(-t)] = 0\}\\
&\quad = \{t: R_{m,n}(-t) + R'_{m,n}(-t) = 0\}\\
&\quad = \{-t: R_{m,n}(t) - R'_{m,n}(t) = 0\}\\
&\quad = \{-t: (d/dt)(1 - \exp(-t)\, R_{m,n}(t)) = 0\}\\
&\quad = \{-t_i\}_{i=1}^{m+n}.
\end{aligned}$$

We now look at the value of the relative error function of S at the $m + n + 2$ points $\{-t_i\}_{i=0}^{m+n+1}$. We have

$$\begin{aligned}
1 - \exp(t_i)\, S(-t_i) &= 1 - (1 - \epsilon_{m,n}^2)/[R_{m,n}(t_i) \exp(-t_i)]\\
&= 1 - (1 - \epsilon_{m,n}^2)/[1 - (-1)^{i+k}\, \epsilon_{m,n}]\\
&= 1 - (1 + (-1)^{i+k}\, \epsilon_{m,n})\\
&= (-1)^{i+k+1}\, \epsilon_{m,n}\,.
\end{aligned}$$

Since the interval $[-h, h]$ is symmetric around the origin, the points $\{-t_i\}_{i=0}^{m+n+1}$ are also in $[-h, h]$. Therefore S satisfies the alternation property, and must be equal to $R_{n,m}$. ∎

COROLLARY 1.

$$\epsilon_{n,m} = \epsilon_{m,n}\,.$$

COROLLARY 2. *If $m = n$, the points $\{t_i\}_{i=0}^{2m+1}$ are symmetrically disposed around the origin, and there exists a polynomial P_m such that*

$$R_{m,m}(t) = (1 - \epsilon_{m,m}^2)^{1/2}\, P_m(t)/P_m(-t). \tag{3}$$

Proof. Let $R_{m,m}(t) = P_{m,m}(t)/Q_{m,m}(t)$. By the theorem,

$$P_{m,m}(-t)\, P_{m,m}(t)/[Q_{m,m}(-t)\, Q_{m,m}(t)] = 1 - \epsilon_{m,m}^2\,. \tag{4}$$

Since $P_{m,m}(t)$ and $Q_{m,m}(t)$ have no common factors, there must be a constant c such that $P_{m,m}(t) = cQ_{m,m}(-t)$ and hence $P_{m,m}(-t) = cQ_{m,m}(t)$. By (4), $c^2 = 1 - \epsilon_{m,m}^2$, and by putting $P_m(t) = Q_{m,m}(-t)$, the corollary follows. ∎

Remark. Once $R_{m,n}^{(-h,h)}$ is known, it is easy to find the relatively best (m, n) approximation $R_{m,n}^{(a,a+2h)}$ to $\exp(x)$ over an interval $[a, a + 2h]$ by

$$R_{m,n}^{(a,a+2h)}(t) = \exp(a + h)\, R_{m,n}^{(-h,h)}(t - a - h)$$

since

$$1 - \exp(-t)\, R_{m,n}^{(a,a+2h)}(t) = 1 - \exp(a + h - t)\, R_{m,n}^{(-h,h)}(t - a - h)$$
$$= 1 - \exp(-x)\, R_{m,n}^{(-h,h)}(x),$$

where $x = t - a - h$ is in $[-h, h]$. This shifting property is shared by Padé approximations, but not by ordinary Chebyshev approximations.

Reference

1. N. Achieser, "Theory of Approximation," Ungar, New York, 1956.